

**ON THE PROPAGATION
OF ELECTROMAGNETIC WAVES IN MEDIA
WITH RELAXATION PHENOMENA**

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Abstract

The propagation and damping of electromagnetic waves in isotropic media with electric conductivity and magnetic relaxation phenomena are investigated. Relaxation phenomena are analyzed using a dynamical constitutive equation obtained by some authors with the help of non-equilibrium thermodynamics. This equation, which generalizes Snoek's equation, has the form of a linear relation among the magnetic induction \mathbf{B} and its first derivative with respect to time, the magnetization \mathbf{M} , the first and second derivatives with respect to time of \mathbf{M} . We derive solutions of the relaxation equation which also satisfy Maxwell equation and using the Laplace transform we show as the solution obtained generalize the classical results.

1. Introduction.

In some papers [1-5] magnetic relaxation phenomena were discussed from the point of view of non-equilibrium

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thermodynamics. Assuming that a vectorial internal variable \mathbf{Z} influences the magnetization, it has been shown that the specific entropy s depends on the specific internal energy u , the tensor of total strain $\varepsilon_{\alpha\beta}$, the specific magnetization \mathbf{m} and the internal variable:

$$(1.1) \quad s = s(u, \varepsilon_{\alpha\beta}, \mathbf{m}, \mathbf{Z})$$

The irreversible magnetic field $\mathbf{B}^{(ir)}$ is defined by

$$(1.2) \quad \mathbf{B}^{(ir)} = \mathbf{B} - \mathbf{B}^{(eq)}$$

The specific magnetization \mathbf{m} is split into two parts

$$(1.3) \quad \mathbf{m} = \mathbf{m}^{(0)} + \mathbf{m}^{(1)}$$

where both changes in $\mathbf{m}^{(0)}$ and $\mathbf{m}^{(1)}$ are of irreversible nature.

Within the usual procedures of the irreversible processes thermodynamics, the following phenomenological equations for magnetic relaxation phenomena in isotropic media [1] are obtained

$$(1.4) \quad \mathbf{B}^{(ir)} = \rho L_{(M)}^{(0,0)} \frac{d\mathbf{m}}{dt} + L_{(M)}^{(0,1)} \mathbf{B}^{(1)}$$

$$(1.5) \quad \rho \frac{d\mathbf{m}^{(1)}}{dt} = \rho L_{(M)}^{(1,0)} \frac{d\mathbf{m}}{dt} + L_{(M)}^{(1,1)} \mathbf{B}^{(1)}$$

The phenomenological coefficients are assumed to be constant because of the linearization of the theory. Onsanger-Casimir reciprocal relations read:

$$(1.6) \quad L_{(M)}^{(0,1)} = -L_{(M)}^{(1,0)}.$$

The two different types of magnetic induction $\mathbf{B}^{(ir)}$ and $\mathbf{B}^{(1)}$ can be derived from the expressions for the free energy or for the entropy (see [1] for details concerning the derivation).

Within the limits of the linear theory and neglecting cross effects, the following dynamical constitutive equation (relaxation equation) was derived:

$$(1.7) \quad \chi_{(BM)}^{(0)} \mathbf{B} + \frac{d\mathbf{B}}{dt} = \chi_{(MB)}^{(0)} \mathbf{M} + \chi_{(MB)}^{(1)} \frac{d\mathbf{M}}{dt} + \chi_{(MB)}^{(2)} \frac{d^2\mathbf{M}}{dt^2}$$

where $\chi_{(BM)}^{(0)}$, $\chi_{(MB)}^{(0)}$, $\chi_{(MB)}^{(1)}$ and $\chi_{(MB)}^{(2)}$ are algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state [1].

Using the relation $\mathbf{M} = \mathbf{B} - \mathbf{H}$ for the magnetization vector, and considering the media at rest, the dynamical constitutive equation (1.7) takes the form:

$$(1.8) \quad \begin{aligned} \chi_{(HB)}^{(0)} \mathbf{H} + \chi_{(HB)}^{(1)} \frac{\partial \mathbf{H}}{\partial t} + \chi_{(HB)}^{(2)} \frac{\partial^2 \mathbf{H}}{\partial t^2} = \chi_{(BH)}^{(0)} \mathbf{B} + \\ + \chi_{(BH)}^{(1)} \frac{\partial \mathbf{B}}{\partial t} + \chi_{(BH)}^{(2)} \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned}$$

In (1.8) the following positions are made

$$(1.9) \quad \begin{aligned} \chi_{(HB)}^{(0)} &= \chi_{(MB)}^{(0)}; \quad \chi_{(HB)}^{(1)} = \chi_{(MB)}^{(1)}; \quad \chi_{(HB)}^{(2)} = \chi_{(MB)}^{(2)} = \chi^{(2)}; \\ \chi_{(BH)}^{(0)} &= \chi_{(MB)}^{(0)} - \chi_{(BM)}^{(0)}; \quad \chi_{(BH)}^{(1)} = \chi_{(MB)}^{(1)} - 1 \end{aligned}$$

Coefficients which occur in (1.9) satisfy the following inequalities, derived [1] from thermodynamics consideration

$$(1.10) \quad \chi_{(HB)}^{(0)} \geq 0, \quad \chi_{(BH)}^{(0)} \geq 0, \quad \chi_{(BH)}^{(2)} = \chi^{(2)} \geq 0,$$

$$(1.11) \quad \chi_{(HB)}^{(1)} > 0, \quad \chi_{(BH)}^{(1)} \geq 0,$$

$$(1.12) \quad \chi_{(BH)}^{(0)} \chi_{(HB)}^{(1)} - \chi_{(HB)}^{(0)} \chi_{(BH)}^{(1)} \geq 0,$$

$$(1.13) \quad \chi_{(HB)}^{(1)} \chi_{(BH)}^{(1)} - \chi^{(2)} \left(\chi_{(HB)}^{(0)} + \chi_{(BH)}^{(0)} \right) > 0.$$

2. Field equations.

The field vectors satisfy Maxwell equations

$$(2.1) \quad \operatorname{rot} \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c} \mathbf{j}^{(el)},$$

$$(2.2) \quad \operatorname{div} \mathbf{D} = \rho^{(el)} \cong 0,$$

$$(2.3) \quad \operatorname{rot} \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0,$$

$$(2.4) \quad \operatorname{div} \mathbf{B} = 0.$$

where \mathbf{E} is the electric field strength, \mathbf{D} is the electric displacement field, $\mathbf{j}^{(el)}$ is the electric current density and $\rho^{(el)}$ is the electric charge density.

Neglecting dielectric relaxation phenomena, the following relation for the electric displacement \mathbf{D} holds

$$(25) \quad \mathbf{D} = \varepsilon \mathbf{E}.$$

If electric conductivity does not vanish, using the method of non-equilibrium thermodynamics, in the case of isotropic media [3], we obtain for the electric current density $\mathbf{j}^{(el)}$

$$(2.6) \quad \mathbf{j}^{(el)} = L_{(P)}^{(el,el)} \mathbf{E};$$

where $L_{(P)}^{(el,el)} \geq 0$.

We look for electromagnetic wave solutions in isotropic media under the conditions given by equations (2.5) and (2.6), and where magnetic relaxation phenomena, governed by the dynamical constitutive equation (1.8), occur. In the following we are considering plane waves in consideration to the fact that, in many practical cases, the factors characterizing the

propagation of these unidimensional fields influence, almost at all, the propagation of more complex fields too.

For plane waves, using a cartesian orthogonal coordinate system $\Sigma \equiv (x, y, z)$ where the z -axis coalesces with the direction of propagation of the wave, we can introduce the following functions for the components of the field vectors \mathbf{E} and \mathbf{H}

$$(2.7) \quad E_x = f(z, t)$$

$$(2.8) \quad H_y = g(z, t)$$

With the condition for the others components

$$(2.9) \quad E_y = H_x = E_z = H_z = 0$$

Equation (2.1) is not identically zero only regarding to the second component and, using notations (2.7)-(2.9), it reads

$$(2.10) \quad \frac{\partial g(z, t)}{\partial z} + \frac{\varepsilon}{c} \frac{\partial f(z, t)}{\partial t} + L_{(P)}^{(el, el)} f(z, t) = 0$$

Analogously, equation (2.3) gives the following results

$$(2.11) \quad B_x = \text{const.}$$

$$(2.12) \quad \frac{\partial B_y}{\partial t} = -c \frac{\partial f(z, t)}{\partial z}$$

$$(2.13) \quad B_z = \text{const.}$$

Introducing (2.11) and (2.13) into the relaxation equation (1.8), we obtain

$$(2.14) \quad B_x = B_z = 0$$

With the position

$$(2.15) \quad B_y = h(z, t)$$

and using the positions (2.8) and (2.15), equation (1.8) takes the form

$$(2.16) \quad \begin{aligned} \chi_{(HB)}^{(0)} g + \chi_{(HB)}^{(1)} \frac{\partial g}{\partial t} + \chi^{(2)} \frac{\partial^2 g}{\partial t^2} = \\ = \chi_{(BH)}^{(0)} h + \chi_{(BH)}^{(1)} \frac{\partial h}{\partial t} + \chi^{(2)} \frac{\partial^2 h}{\partial t^2} \end{aligned}$$

2. Wave solutions.

We obtain solutions for the problem given by the relation (2.10), (2.12) and (2.16) in the case of specific boundary conditions, using the method of Laplace transform.

The electromagnetic field is taken zero at each point at the initial time and this condition is given by

$$(3.1) \quad h(z, 0) = 0, \quad \left(\frac{\partial h(z, t)}{\partial t} \right)_{t=0} = 0$$

and

$$(3.2) \quad f(z, 0) = 0, \quad \left(\frac{\partial f(z, t)}{\partial t} \right)_{t=0} = 0.$$

We use the following position to indicate the Laplace transform of the functions considered with respect to the variable t , indicating with ξ the transformation variable

$$(3.3) \quad F(z, \xi) = L[f(z, t)]$$

$$(3.4) \quad G(z, \xi) = L[g(z, t)]$$

$$(3.5) \quad H(z, \xi) = L[h(z, t)]$$

Using the initial conditions (3.1) and (3.2) and operating the Laplace transform, equations (2.10), (2.12) and (2.16) take the form

$$(3.6) \quad \frac{\partial G(z, \xi)}{\partial z} + \left(\varepsilon \xi + L_{(P)}^{(el,le)} \right) F(z, \xi) = 0$$

$$(3.7) \quad H(z, \xi) = -\frac{c}{\xi} \frac{\partial F(z, \xi)}{\partial z}$$

$$(3.8) \quad H(z, \xi) = \frac{\chi^{(2)} \xi^2 + \chi_{(HB)}^{(1)} \xi + \chi_{(HB)}^{(0)}}{\chi^{(2)} \xi^2 + \chi_{(BH)}^{(1)} \xi + \chi_{(BH)}^{(0)}} G(z, \xi)$$

By solving the system of equations (3.6)-(3.8), we obtain the following ordinary differential equation

$$(3.9) \quad \frac{d^2 F}{dz^2} - \frac{\xi(\varepsilon + L_{(P)}^{(el,el)})}{c^2} \cdot \frac{\chi^{(2)} \xi^2 + \chi_{(HB)}^{(1)} \xi + \chi_{(HB)}^{(0)}}{\chi^{(2)} \xi^2 + \chi_{(BH)}^{(1)} \xi + \chi_{(BH)}^{(0)}} F(z, \xi) = 0$$

Considering the inequalities (1.10)-(1.13) about the phenomenological coefficients, we note that the parameter ξ of the Laplace transform is real in the case that

$$(3.10) \quad \chi_{(HB)}^{(1)} = \chi_{(BH)}^{(1)}.$$

Using the following position

$$(3.11) \quad m^2 = \frac{\xi(\varepsilon \xi + L_{(P)}^{(el,el)})}{c^2} \frac{\chi^{(2)} \xi^2 + \chi_{(HB)}^{(1)} \xi + \chi_{(HB)}^{(0)}}{\chi^{(2)} \xi^2 + \chi_{(BH)}^{(1)} \xi + \chi_{(BH)}^{(0)}}$$

we obtain the following solution of the differential equation (3.9)

$$(3.12) \quad F(z, \xi) = Ae^{mz} + Be^{-mz}$$

where the coefficients A and B are specified by boundary conditions given in the following.

We consider electromagnetic wave travelling along the positive direction of the z -axis, with initial intensity given by

$$(3.13) \quad f(0, t) = f_3(t)$$

Using the same formalism as before about the laplace transform of the initial intensity, by (3.13) and (3.12) we obtain the following solution

$$(3.14) \quad F(z, \xi) = F_3(\xi)e^{-mz}$$

Equation (3.14) represents a generalization of the classical solution for electromagnetic waves into an isotropic medium with electric conductivity. In equilibrium conditions the following relation holds

$$(3.15) \quad \mathbf{B} = \mu_{(eq)} \mathbf{H}$$

where the magnetic permeability, at the equilibrium, is given, in terms the coefficients occurring in the relaxation equation (1.8) such that

$$(3.16) \quad \mu_{(eq)} = \frac{\chi_{(HB)}^{(0)}}{\chi_{(BH)}^{(0)}} \quad \text{with} \quad \chi^{(2)} = \chi_{(HB)}^{(1)} = \chi_{(BH)}^{(1)} = 0$$

In this particular case, from equation (3.14), making the positions

$$(3.17) \quad L_{(P)}^{(el,el)} \quad \text{and} \quad \mu = \mu_{(eq)}$$

we obtain the wave solution

$$(3.18) \quad f(z, t) = e^{-\frac{\sigma}{2\varepsilon}\sqrt{\mu\varepsilon}z} f_3(t - \sqrt{\mu\varepsilon}z) = \\ - \frac{1}{\sqrt{\mu\varepsilon}} \int_{\sqrt{\mu\varepsilon}z}^t f_3(t - \tau) e^{-\frac{\sigma}{2\varepsilon}\tau} \frac{\partial}{\partial z} \mathbf{J}_0(\alpha) d\tau$$

where \mathbf{J}_0 is the Bessel function of order 0 and argument α given by

$$(3.19) \quad \alpha = \frac{\sigma}{2\varepsilon} \sqrt{z^2 + \frac{1}{\mu\varepsilon}\tau^2}$$

We notice that equation (3.18) represents classical wave solution [11] when magnetic relaxation phenomena are not taken into account.

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